A COUNTEREXAMPLE TO WEIGHTED ESTIMATES FOR MULTILINEAR FOURIER MULTIPLIERS WITH SOBOLEV REGULARITY

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Abstract. The problem whether weighted estimates for multilinear Fourier multipliers with Sobolev regularity hold under weak condition on weights is considered.

1. Introduction

In this paper, we consider weighted norm inequalities for multilinear Fourier multipliers with Sobolev regularity. Before discussing them, we briefly recall some basic facts on weights in the multilinear theory.

In the linear case, it is well known that the Hardy-Littlewood maximal operator M is bounded on $L^p(w)$ if and only if the weight w belongs to the Muckenhoupt class A_p, where $1 < p < \infty$ and M is defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy$$

for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ (see Section 2 for the definitions not given in this section). Let $1 < p_1, \dots, p_N < \infty$ and $1/p_1 + \cdots + 1/p_N = 1/p$. For $\vec{w} = (w_1, \dots, w_N)$, we set $\nu_{\vec{w}} = \prod_{k=1}^N w_k^{p/p_k}$. By Hölder's inequality, if $\vec{w} = (w_1, \dots, w_N) \in A_{p_1} \times \cdots \times A_{p_N}$, then

$$\left\| \prod_{k=1}^N M f_k \right\|_{L^p(\nu_{\vec{w}})} \lesssim \prod_{k=1}^N \|f_k\|_{L^{p_k}(w_k)}. \tag{1.1}$$

We define the multi(sub)linear maximal operator $\mathcal{M}$ by

$$\mathcal{M}(\vec{f})(x) = \sup_{Q \ni x} \prod_{k=1}^N \frac{1}{|Q|} \int_Q |f_k(y_k)|\, dy_k$$

for $\vec{f} = (f_1, \dots, f_N) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)^N$, and note that

$$\mathcal{M}(\vec{f})(x) \le \prod_{k=1}^N M f_k(x).$$

Lerner, Ombrosi, Pérez, Torres and Trujillo-González [12] proved that $\mathcal{M}$ is bounded from $L^{p_1}(w_1) \times \cdots \times L^{p_N}(w_N)$ to $L^p(\nu_{\vec{w}})$ if and only if $\vec{w} \in A_{(p_1,\dots,p_N)}$. It should

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be remarked that there exists $\vec{w} \in A_{(p_1,\dots,p_N)}$ such that (1.1) does not hold ([12, Remark 7.5]). This says that the inclusion

$$A_{p_1} \times \cdots \times A_{p_N} \subset A_{(p_1,\dots,p_N)} \tag{1.2}$$

is strict.

Let $m \in L^\infty(\mathbb{R}^{Nn})$. The N-linear Fourier multiplier operator T_m is defined by

$$T_m(\vec{f})(x) = \frac{1}{(2\pi)^{Nn}} \int_{\mathbb{R}^{Nn}} e^{ix\cdot(\xi_1+\cdots+\xi_N)} m(\xi) \widehat{f_1}(\xi_1) \dots \widehat{f_N}(\xi_N)\, d\xi$$

for $\vec{f} = (f_1, \dots, f_N) \in \mathcal{S}(\mathbb{R}^n)^N$, where $x \in \mathbb{R}^n$, $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ and $d\xi = d\xi_1, \dots, d\xi_N$. It is well known that in the unweighted case the boundedness of T_m holds if

$$|\partial_\xi^\alpha m(\xi)| \le C_\alpha |\xi|^{-|\alpha|}$$

for sufficiently many multi-indices α (see, for example, [2, 7, 10] and also [5, 6, 15, 16] for multipliers with Sobolev regularity). We set

$$m_j(\xi) = m(2^j\xi)\Psi(\xi), \qquad j \in \mathbb{Z}, \tag{1.3}$$

where Ψ is a function in $\mathcal{S}(\mathbb{R}^{Nn})$ satisfying

$$\operatorname{supp}\Psi \subset \{\xi \in \mathbb{R}^{Nn} : 1/2 \le |\xi| \le 2\}, \qquad \sum_{k\in\mathbb{Z}} \Psi(\xi/2^k) = 1,\ \xi \in \mathbb{R}^{Nn} \setminus \{0\}.$$

We use the notation $\|T_m\|_{L^{p_1}(w_1)\times\cdots\times L^{p_N}(w_N)\to L^p(\nu_{\vec{w}})}$ to denote the smallest constant C satisfying

$$\|T_m(\vec{f})\|_{L^p(\nu_{\vec{w}})} \le C \prod_{k=1}^N \|f_k\|_{L^{p_k}(w_k)}$$

for all $\vec{f} = (f_1, \dots, f_N) \in \mathcal{S}(\mathbb{R}^n)^N$.

Let $Nn/2 < s \le Nn$, $Nn/s < p_1, \dots, p_N < \infty$ and $1/p_1 + \cdots + 1/p_N = 1/p$. It follows from [3, Theorem 6.2] that if $\vec{w} = (w_1, \dots, w_N) \in A_{p_1 s/(Nn)} \times \cdots \times A_{p_N s/(Nn)}$, then

$$\|T_m\|_{L^{p_1}(w_1)\times\cdots\times L^{p_N}(w_N)\to L^p(\nu_{\vec{w}})} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(s/N,\dots,s/N)}(\mathbb{R}^{Nn})}, \tag{1.4}$$

where the implicit constant is independent of m (see Li, Xue and Yabuta [14] for the endpoint cases). This result can also be obtained from another approach of Hu and Lin [8]. Replacing $W^{(s/N,\dots,s/N)}$ by W^s, Bui and Duong [1], Li and Sun [13] proved that if $\vec{w} = (w_1, \dots, w_N) \in A_{(p_1 s/(Nn),\dots,p_N s/(Nn))}$, then

$$\|T_m\|_{L^{p_1}(w_1)\times\cdots\times L^{p_N}(w_N)\to L^p(\nu_{\vec{w}})} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{W^s(\mathbb{R}^{Nn})}. \tag{1.5}$$

By the embedding

$$W^s(\mathbb{R}^{Nn}) \hookrightarrow W^{(s/N,\dots,s/N)}(\mathbb{R}^{Nn}),$$

we note that the regularity condition in (1.5) is stronger than that in (1.4). Of course, it follows from (1.2) that estimate (1.5) holds if $\vec{w} = (w_1, \dots, w_N) \in A_{p_1 s/(Nn)} \times \cdots \times A_{p_N s/(Nn)}$. See Table 1 for the three cases mentioned here. If $N = 1$ (namely, the linear case), estimate (1.4) is the same as (1.5), and due to Kurtz and Wheeden [11].

The purpose of this paper is to answer the question whether estimate (1.4) holds under the condition $\vec{w} = (w_1, \dots, w_N) \in A_{(p_1 s/(Nn),\dots,p_N s/(Nn))}$, and the main result is the following:

	$A_{p_1 s/(Nn)} \times \cdots \times A_{p_N s/(Nn)}$	$A_{(p_1 s/(Nn),\dots,p_N s/(Nn))}$
$W^s(\mathbb{R}^{Nn})$	hold	hold
$W^{(s/N,\dots,s/N)}(\mathbb{R}^{Nn})$	hold	?

TABLE 1. Weighted estimates from $L^{p_1}(w_1) \times \cdots \times L^{p_N}(w_N)$ to $L^p(\nu_{\vec{w}})$.

Theorem 1.1. *Let $N \geq 2$, $Nn/2 < s \leq Nn$, $Nn/s < p_1, \dots, p_N < \infty$ and $1/p_1 + \cdots + 1/p_N = 1/p$. Then there exists $\vec{w} = (w_1, \dots, w_N) \in A_{(p_1 s/(Nn),\dots,p_N s/(Nn))}$ such that the estimate*

$$\|T_m\|_{L^{p_1}(w_1)\times\cdots\times L^{p_N}(w_N)\to L^p(\nu_{\vec{w}})} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(s/N,\dots,s/N)}(\mathbb{R}^{Nn})}$$

does not hold, where the implicit constant is independent of m.

It should be pointed out that the statement similar to Theorem 1.1 holds even if we replace $L^p(\nu_{\vec{w}})$ by $L^{p,\infty}(\nu_{\vec{w}})$ (see Remark 3.1). Using the class $A_{\vec{P}/\vec{Q}}$ which coincides with $A_{(p_1 s/(Nn),\dots,p_N s/(Nn))}$ if $\vec{P} = (p_1, \dots, p_N)$ and $\vec{Q} = (Nn/s, \dots, Nn/s)$, Jiao [9] gave a generalization of (1.5). See Remark 3.2 for the result corresponding to this weight class.

2. Preliminaries

For two non-negative quantities A and B, the notation $A \lesssim B$ means that $A \leq CB$ for some unspecified constant $C > 0$, and $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$. For $1 < p < \infty$, p' is the conjugate exponent of p, that is, $1/p + 1/p' = 1$.

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of all rapidly decreasing smooth functions. We define the Fourier transform $\mathcal{F}f$ and the inverse Fourier transform $\mathcal{F}^{-1}f$ of $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx \quad \text{and} \quad \mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)\, d\xi.$$

The (usual) Sobolev space $W^s(\mathbb{R}^{Nn})$, $s \in \mathbb{R}$, is defined by the norm

$$\|F\|_{W^s(\mathbb{R}^{Nn})} = \left(\int_{\mathbb{R}^{Nn}} (1 + |\xi|^2)^s |\widehat{F}(\xi)|^2\, d\xi \right)^{1/2},$$

where $\widehat{F}$ is the Fourier transform in all the variables. The Sobolev space of product type $W^{(s_1,\dots,s_N)}(\mathbb{R}^{Nn})$, $s_1, \dots, s_N \in \mathbb{R}$, is also defined by the norm

$$\|F\|_{W^{(s_1,\dots,s_N)}(\mathbb{R}^{Nn})} = \left(\int_{\mathbb{R}^{Nn}} (1 + |\xi_1|^2)^{s_1} \dots (1 + |\xi_N|^2)^{s_N} |\widehat{F}(\xi)|^2\, d\xi \right)^{1/2},$$

where $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n$.

Let $w \geq 0$. For a measurable set E, we write $w(E) = \int_E w(x)\, dx$, and simply $|E| = \int_E dx$ for the case $w \equiv 1$. The weighted Lebesgue space $L^p(w)$, $0 < p < \infty$, consists of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx \right)^{1/p} < \infty.$$

The weighted weak Lebesgue space $L^{p,\infty}(w)$ is also defined by the norm

$$\|f\|_{L^{p,\infty}(w)} = \sup_{\lambda>0} \lambda \left\{ w\left(\{x \in \mathbb{R}^n : |f(x)| > \lambda\} \right) \right\}^{1/p}.$$

We say that a weight w belongs to the Muckenhoupt class A_p, $1 < p < \infty$, if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q w(x)\, dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{1-p'}\, dx \right)^{p-1} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the axes. We also say that $\vec{w} = (w_1, \dots, w_N)$ belongs to the class $A_{(p_1,\dots,p_N)}$, $1 < p_1, \dots, p_N < \infty$, if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q \nu_{\vec{w}}(x)\, dx \right)^{1/p} \prod_{k=1}^{N} \left(\frac{1}{|Q|} \int_Q w_k(x)^{1-p_k'}\, dx \right)^{1/p_k'} < \infty,$$

where $1/p_1 + \cdots + 1/p_N = 1/p$ and $\nu_{\vec{w}} = \prod_{k=1}^N w_k^{p/p_k}$.

The proof of the following lemma is based on the argument of [4, Example 9.1.7].

Lemma 2.1. *Let $N \geq 2$, $1 < p_1, \dots, p_N < \infty$ and $1/p_1 + \cdots + 1/p_N = 1/p$. If α_1, α_2 satisfy $\alpha_1/p_1 + \alpha_2/p_2 > -n/p$ and $\alpha_k < n(p_k - 1)$ for $k = 1, 2$, then*

$$\vec{w} = (w_1, w_2, w_3, w_4, \dots, w_N) = (|x|^{\alpha_1}, |x|^{\alpha_2}, 1, 1, \dots, 1)$$

belongs to the class $A_{(p_1,\dots,p_N)}$.

Proof. Since $w_k = 1$ for $k \geq 3$, the desired conclusion follows from

$$\sup_B \left(\frac{1}{|B|} \int_B w_1^{p/p_1} w_2^{p/p_2}\, dx \right)^{1/p} \prod_{k=1}^{2} \left(\frac{1}{|B|} \int_B w_k^{1-p_k'}\, dx \right)^{1/p_k'} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$ (instead of cubes). Let B be the ball with center x_0 and radius r.

We first consider the case $|x_0| \geq 2r$. In this case, $|x| \approx |x_0|$ for all $x \in B$. Then

$$\begin{aligned} &\left(\frac{1}{|B|} \int_B |x|^{p(\alpha_1/p_1 + \alpha_2/p_2)}\, dx \right)^{1/p} \prod_{k=1}^{2} \left(\frac{1}{|B|} \int_B |x|^{\alpha_k(1-p_k')}\, dx \right)^{1/p_k'} \\ &\approx |x_0|^{\alpha_1/p_1 + \alpha_2/p_2} |x_0|^{\alpha_1(1/p_1' - 1)} |x_0|^{\alpha_2(1/p_2' - 1)} = 1. \end{aligned}$$

We next consider the case $|x_0| < 2r$. In this case, $B \subset \{x \in \mathbb{R}^n : |x| < 3r\}$. Since $p(\alpha_1/p_1 + \alpha_2/p_2) > -n$ and $\alpha_k(1 - p_k') > -n$ for $k = 1, 2$, we have

$$\int_B |x|^{p(\alpha_1/p_1 + \alpha_2/p_2)}\, dx \leq \int_{|x|<3r} |x|^{p(\alpha_1/p_1 + \alpha_2/p_2)}\, dx \lesssim r^{p(\alpha_1/p_1 + \alpha_2/p_2) + n}$$

and

$$\int_B |x|^{\alpha_k(1-p_k')}\, dx \leq \int_{|x|<3r} |x|^{\alpha_k(1-p_k')}\, dx \lesssim r^{\alpha_k(1-p_k') + n}, \qquad k = 1, 2.$$

Hence,

$$\begin{aligned} &\left(\frac{1}{|B|} \int_B |x|^{p(\alpha_1/p_1 + \alpha_2/p_2)}\, dx \right)^{1/p} \prod_{k=1}^{2} \left(\frac{1}{|B|} \int_B |x|^{\alpha_k(1-p_k')}\, dx \right)^{1/p_k'} \\ &\lesssim r^{\alpha_1/p_1 + \alpha_2/p_2} r^{\alpha_1(1/p_1' - 1)} r^{\alpha_2(1/p_2' - 1)} = 1. \end{aligned}$$

The proof is complete. □

The following fact is known, but we shall give a proof for the reader's convenience.

Lemma 2.2. *Let $r > 0$, and let ℓ be a non-negative integer. Then there is a function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ so that $\operatorname{supp}\varphi \subset \{x \in \mathbb{R}^n : |x| \le r\}$, $\int_{\mathbb{R}^n} \varphi(x)^2\,dx \neq 0$ and $\int_{\mathbb{R}^n} x^\beta \varphi(x)\,dx = 0$ for all multi-indices β satisfying $|\beta| \le \ell$.*

Proof. Let $\psi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ be a real valued function satisfying $\operatorname{supp}\psi \subset \{x \in \mathbb{R}^n : |x| \le r\}$, and set $\varphi(x) = (-\Delta)^{\ell+1}\psi(x)$, where $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$.

We shall check that φ satisfies all the required conditions. Obviously, $\operatorname{supp}\varphi \subset \{x \in \mathbb{R}^n : |x| \le r\}$. Since ψ is not identically equal to zero, so is $\widehat{\psi}$. Thus, we can take $\xi_0 \in \mathbb{R}^n$ and $r_0 > 0$ such that $\widehat{\psi}(\xi) \neq 0$ if $|\xi - \xi_0| \le r_0$. Since φ is a real valued function, we have by Plancherel's theorem

$$\begin{aligned}\int_{\mathbb{R}^n} \varphi(x)^2\,dx &= \int_{\mathbb{R}^n} \left|(-\Delta)^{\ell+1}\psi(x)\right|^2\,dx = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \left||\xi|^{2(\ell+1)}\widehat{\psi}(\xi)\right|^2\,d\xi \\ &\ge \frac{1}{(2\pi)^n}\int_{|\xi-\xi_0|\le r_0} \left||\xi|^{2(\ell+1)}\widehat{\psi}(\xi)\right|^2\,d\xi \gtrsim \int_{|\xi-\xi_0|\le r_0} |\xi|^{4(\ell+1)}\,d\xi \neq 0.\end{aligned}$$

Finally,

$$\int_{\mathbb{R}^n} (-ix)^\beta \varphi(x)\,dx = \partial^\beta \widehat{\varphi}(\xi)\Big|_{\xi=0} = \partial^\beta \left(|\xi|^{2(\ell+1)}\widehat{\psi}(\xi)\right)\Big|_{\xi=0} = 0$$

for $|\beta| \le \ell$. This completes the proof. $\square$

3. Proof of Theorem 1.1

In this section, using the ideas given in [5, 15, Section 7] and [12, Remark 7.5], we shall prove Theorem 1.1.

Proof of Theorem 1.1. Let $N \ge 2$, $Nn/2 < s \le Nn$, $Nn/s < p_1, \dots, p_N < \infty$ and $1/p_1 + \cdots + 1/p_N = 1/p$. We first claim that there exist $\alpha_1 < -n$ and $\alpha_2 > -n$ such that

$$\alpha_1/p_1 + \alpha_2/p_2 > -n/p, \qquad \alpha_k/p_k < s/N - n/p_k, \quad k = 1, 2, \tag{3.1}$$

and

$$\alpha_1/p_1 < -n/p_1 - s/N + n/2. \tag{3.2}$$

Indeed, since $-n/p + n/p_1 + s/N - n/2 < s/N - n/p_2$ and $s/N - n/p_2 > 0$, we can take $\alpha_2 \ge 0$ satisfying $-n/p + n/p_1 + s/N - n/2 < \alpha_2/p_2 < s/N - n/p_2$. Then, since $-\alpha_2/p_2 - n/p < -n/p_1 - s/N + n/2$, we can take α_1 satisfying $-\alpha_2/p_2 - n/p < \alpha_1/p_1 < -n/p_1 - s/N + n/2$. It is easy to check that these α_1, α_2 satisfy $\alpha_1 < -n$, $\alpha_2 > -n$, (3.1) and (3.2).

For $\alpha_1 < -n$ and $\alpha_2 > -n$ satisfying (3.1) and (3.2), we set

$$\vec{w} = (w_1, w_2, w_3, w_4, \dots, w_N) = (|x|^{\alpha_1}, |x|^{\alpha_2}, 1, 1, \dots, 1). \tag{3.3}$$

Let $(q_1, \dots, q_N) = (p_1 s/(Nn), \dots, p_N s/(Nn))$ and $1/q_1 + \cdots + 1/q_N = 1/q$. Since $p/p_k = q/q_k$ for $k = 1, 2$, it follows from (3.1) that $\alpha_1/q_1 + \alpha_2/q_2 > -n/q$ and $\alpha_k < n(q_k - 1)$ for $k = 1, 2$. Then, by Lemma 2.1, we see that $\vec{w} \in A_{(q_1,\dots,q_N)}$.

We shall prove Theorem 1.1 with $\vec{w}$ defined by (3.1), (3.2) and (3.3) by contradiction. To do this, we assume that the estimate

$$\|T_m\|_{L^{p_1}(w_1)\times\cdots\times L^{p_N}(w_N)\to L^p(\nu_{\vec{w}})} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(s/N,\dots,s/N)}} \tag{3.4}$$

holds, where the implicit constant is independent of m. Let $\widehat{\varphi} \in \mathcal{S}(\mathbb{R}^n)$ be a function as in Lemma 2.2 with $r = 1/(10N)$ and ℓ satisfying $p_1(\ell+1) + \alpha_1 > -n$: $\operatorname{supp} \widehat{\varphi} \subset \{\eta \in \mathbb{R}^n : |\eta| \le 1/(10N)\}$,

$$\int_{\mathbb{R}^n} \widehat{\varphi}(\eta)^2 \, d\eta \neq 0, \tag{3.5}$$

$$\int_{\mathbb{R}^n} \eta^\beta \widehat{\varphi}(\eta) \, d\eta = 0, \quad |\beta| \le \ell. \tag{3.6}$$

For sufficiently small $\epsilon > 0$, we set

$$m^{(\epsilon)}(\xi) = \widehat{\varphi}((\xi_1 - e_1)/\epsilon)\widehat{\varphi}(\xi_2)\widehat{\varphi}(\xi_3) \times \cdots \times \widehat{\varphi}(\xi_N), \tag{3.7}$$

where $e_1 = (1, 0, 0, \ldots, 0) \in \mathbb{R}^n$.

We shall estimate the Sobolev norm of $m^{(\epsilon)}$ as follows:

$$\sup_{j \in \mathbb{Z}} \|m_j^{(\epsilon)}\|_{W^{(s/N, \ldots, s/N)}} \lesssim \epsilon^{-s/N + n/2}, \tag{3.8}$$

where $m_j^{(\epsilon)}$ is defined by (1.3) with m replaced by $m^{(\epsilon)}$. To do this, we choose the function $\Psi \in \mathcal{S}(\mathbb{R}^{Nn})$ appearing in the definition of $m_j^{(\epsilon)}$ so that

$$\begin{aligned} &\operatorname{supp} \Psi \subset \{\xi \in \mathbb{R}^{Nn} : 2^{-1/2-\gamma} \le |\xi| \le 2^{1/2+\gamma}\}, \\ &\Psi(\xi) = 1 \quad \text{if} \quad 2^{-1/2+\gamma} \le |\xi| \le 2^{1/2-\gamma}, \end{aligned}$$

where $\gamma > 0$ is a sufficiently small number. If $\epsilon > 0$ is sufficiently small, then

$$\begin{aligned} \operatorname{supp} m^{(\epsilon)} &\subset \{(\xi_1, \ldots, \xi_N) : |\xi_1 - e_1| \le \epsilon/(10N),\ |\xi_k| \le 1/(10N),\ 2 \le k \le N\} \\ &\subset \{(\xi_1, \ldots, \xi_N) : 2^{-1/2+\gamma} \le (|\xi_1|^2 + \cdots + |\xi_N|^2)^{1/2} \le 2^{1/2-\gamma}\}. \end{aligned}$$

This implies

$$m_j^{(\epsilon)}(\xi) = m^{(\epsilon)}(2^j \xi)\Psi(\xi) = \begin{cases} m^{(\epsilon)}(\xi) & \text{if} \quad j = 0 \\ 0 & \text{if} \quad j \neq 0, \end{cases}$$

and consequently

$$\sup_{j \in \mathbb{Z}} \|m_j^{(\epsilon)}\|_{W^{(s/N, \ldots, s/N)}} = \|m^{(\epsilon)}\|_{W^{(s/N, \ldots, s/N)}} = \|\widehat{\varphi}((\cdot - e_1)/\epsilon)\|_{W^{s/N}} \|\widehat{\varphi}\|_{W^{s/N}}^{N-1}.$$

Taking a sufficiently large $L > 0$, we have

$$\begin{aligned} &\|\widehat{\varphi}((\cdot - e_1)/\epsilon)\|_{W^{s/N}} = (2\pi)^n \|(1 + |\cdot|^2)^{s/(2N)} \epsilon^n \varphi(\epsilon\, \cdot)\|_{L^2} \\ &\lesssim \epsilon^n \left(\int_{\mathbb{R}^n} (1 + |x|)^{2s/N} (1 + \epsilon|x|)^{-2L} \, dx \right)^{1/2} \\ &\lesssim \epsilon^n \left(\int_{|x| \le 1} dx + \int_{1 < |x| \le 1/\epsilon} |x|^{2s/N} \, dx + \int_{|x| > 1/\epsilon} |x|^{2s/N} (\epsilon|x|)^{-2L} \, dx \right)^{1/2} \\ &\lesssim \epsilon^{-s/N + n/2}, \end{aligned}$$

and we obtain (3.8).

Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\widehat{\psi} = 1$ on $\operatorname{supp} \widehat{\varphi}$, and set

$$\widehat{f_1}(\xi_1) = \widehat{f_1^{(\epsilon)}}(\xi_1) = \epsilon^{n/p_1 - n} \widehat{\varphi}((\xi_1 - e_1)/\epsilon), \qquad \widehat{f_k}(\xi_k) = \widehat{\psi}(\xi_k),\ 2 \le k \le N. \tag{3.9}$$

To estimate $\|f_1^{(\epsilon)}\|_{L^{p_1}(w_1)}$, we check that φ belongs to $L^{p_1}(w_1)$. It follows from (3.6) that $\partial^\beta \varphi(0) = 0$ for $|\beta| \le \ell$. Combining this with Taylor's formula, we see that $|\varphi(x)| \lesssim |x|^{\ell+1}$. Then, since $p_1(\ell+1) + \alpha_1 > -n$,

$$\int_{|x|<1} |\varphi(x)|^{p_1} |x|^{\alpha_1} dx \lesssim \int_{|x|<1} |x|^{p_1(\ell+1)+\alpha_1} dx < \infty.$$

On the other hand, it is obvious that

$$\int_{|x|\ge 1} |\varphi(x)|^{p_1} |x|^{\alpha_1} dx < \infty.$$

Consequently, φ belongs to $L^{p_1}(w_1)$. Hence,

$$\|f_1^{(\epsilon)}\|_{L^{p_1}(w_1)} = \left(\int_{\mathbb{R}^n} |\epsilon^{n/p_1} \varphi(\epsilon x)|^{p_1} |x|^{\alpha_1} \, dx \right)^{1/p_1} = \epsilon^{-\alpha_1/p_1} \|\varphi\|_{L^{p_1}(w_1)}. \tag{3.10}$$

The condition $\alpha_2 > -n$ implies that $w_2 = |x|^{\alpha_2}$ is locally integrable. Then, since ψ is rapidly decreasing, we have $\|f_2\|_{L^{p_2}(w_2)} = \|\psi\|_{L^{p_2}(|x|^{\alpha_2})} < \infty$ and $\|f_k\|_{L^{p_k}(w_k)} = \|\psi\|_{L^{p_k}} < \infty$ for $k = 3, \dots, N$.

We shall finish the proof. By (3.7) and (3.9),

$$\begin{aligned} T_{m^{(\epsilon)}}(\vec{f})(x) &= \mathcal{F}^{-1}[\widehat{\varphi}((\cdot - e_1)/\epsilon)\widehat{f_1}](x)\mathcal{F}^{-1}[\widehat{\varphi}\widehat{f_2}](x) \dots \mathcal{F}^{-1}[\widehat{\varphi f_N}](x) \\ &= \mathcal{F}^{-1}[\epsilon^{n/p_1 - n}\widehat{\varphi}((\cdot - e_1)/\epsilon)^2](x)\mathcal{F}^{-1}[\widehat{\varphi}](x) \dots \mathcal{F}^{-1}[\widehat{\varphi}](x) \\ &= \epsilon^{n/p_1} e^{ie_1\cdot x}(\varphi * \varphi)(\epsilon x)\varphi(x)^{N-1}, \end{aligned} \tag{3.11}$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform on $\mathbb{R}^n$. Since φ is not identically equal to zero and $\nu_{\vec{w}} = |x|^{p(\alpha_1/p_1+\alpha_2/p_2)}$ is locally integrable (see (3.1)), we can take $R > 0$ satisfying

$$0 < \int_{|x|\le R} |\varphi(x)|^{p(N-1)} \nu_{\vec{w}}(x) \, dx < \infty. \tag{3.12}$$

On the other hand, it follows from (3.5) that

$$\varphi * \varphi(0) = \mathcal{F}^{-1}[\widehat{\varphi}\,\widehat{\varphi}](0) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{\varphi}(\eta)^2 \, d\eta \neq 0.$$

By the continuity of $\varphi * \varphi$ at the origin, there exist $C > 0$ and ϵ_0 such that

$$|\varphi * \varphi(\epsilon x)| \ge C \qquad \text{for all } 0 < \epsilon < \epsilon_0 \text{ and } |x| \le R. \tag{3.13}$$

Thus,

$$\|T_{m^{(\epsilon)}}(\vec{f})\|_{L^p(\nu_{\vec{w}})} \ge C\epsilon^{n/p_1} \left(\int_{|x|\le R} |\varphi(x)|^{p(N-1)} \nu_{\vec{w}}(x) \, dx \right)^{1/p} = C\epsilon^{n/p_1} \tag{3.14}$$

for all $0 < \epsilon < \epsilon_0$. Hence, by (3.4), (3.8) and (3.10),

$$\begin{aligned} \epsilon^{n/p_1} &\lesssim \|T_{m^{(\epsilon)}}(\vec{f})\|_{L^p(\nu_{\vec{w}})} \le \|T_{m^{(\epsilon)}}\|_{L^{p_1}(w_1)\times\cdots\times L^{p_N}(w_N)\to L^p(\nu_{\vec{w}})} \prod_{k=1}^N \|f_k\|_{L^{p_k}(w_k)} \\ &\lesssim \sup_{j\in\mathbb{Z}} \|m_j^{(\epsilon)}\|_{W^{(s/N,\dots,s/N)}} \|f_1^{(\epsilon)}\|_{L^{p_1}(w_1)} \prod_{k=2}^N \|f_k\|_{L^{p_k}(w_k)} \lesssim \epsilon^{-s/N+n/2-\alpha_1/p_1} \end{aligned}$$

for all sufficiently small $\epsilon > 0$. However, since $n/p_1 < -s/N + n/2 - \alpha_1/p_1$ (see (3.2)), this is a contradiction. Therefore, estimate (3.4) does not hold. $\square$

We end this paper by giving the two remarks mentioned in the end of the introduction.

Remark 3.1. Let N, s and $p_1, \dots, p_N$ satisfy the assumption of Theorem 1.1. Once inequality (3.14) is replaced by the sharper one

$$\|T_{m^{(\epsilon)}}(\vec{f})\|_{L^{p,\infty}(\nu_{\vec{w}})} \gtrsim \epsilon^{n/p_1} \qquad \text{for all sufficiently small } \epsilon > 0, \tag{3.15}$$

where $m^{(\epsilon)}$, $\vec{f}$ and $\vec{w}$ are the same as in the proof of Theorem 1.1, the same argument as before shows that the estimate

$$\|T_m\|_{L^{p_1}(w_1)\times\cdots\times L^{p_N}(w_N)\to L^{p,\infty}(\nu_{\vec{w}})} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(s/N,\dots,s/N)}(\mathbb{R}^{Nn})}$$

does not hold, where the implicit constant is independent of m.

It is not difficult to prove (3.15). Indeed, by (3.11) and (3.13),

$$\left\{x \in \mathbb{R}^n : |T_{m^{(\epsilon)}}(\vec{f})(x)| > \lambda\right\} \supset \left\{x \in B_R : |\varphi(x)|^{N-1} > (C\epsilon^{n/p_1})^{-1}\lambda\right\}$$

for all $0 < \epsilon < \epsilon_0$ and $\lambda > 0$, where B_R is the ball with center at the origin and radius R. Hence, since $0 < \sup_{\lambda>0} \lambda \left\{\nu_{\vec{w}}\left(\left\{x \in B_R : |\varphi(x)|^{N-1} > \lambda\right\}\right)\right\}^{1/p} < \infty$ (see (3.12)), we obtain (3.15).

Remark 3.2. Let $1 \le q_k < p_k$, $k = 1, \dots, N$, and set $\vec{P} = (p_1, \dots, p_N)$, $\vec{Q} = (q_1, \dots, q_N)$. We say that $\vec{w} = (w_1, \dots, w_N)$ belongs to the class $A_{\vec{P}/\vec{Q}}$ if

$$\sup_Q \left(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}(x)\,dx\right)^{1/p} \prod_{k=1}^N \left(\frac{1}{|Q|}\int_Q w_k^{1-(p_k/q_k)'}(x)\,dx\right)^{1/q_k - 1/p_k} < \infty,$$

where $1/p_1 + \cdots + 1/p_N = 1/p$ and $\nu_{\vec{w}} = \prod_{k=1}^N w_k^{p/p_k}$. Note that $\prod_{k=1}^N A_{p_k/q_k} \subset A_{\vec{P}/\vec{Q}}$, and $A_{\vec{P}/\vec{Q}} = A_{(p_1/q_0,\dots,p_N/q_0)}$ if $q_1 = \cdots = q_N = q_0 \ge 1$.

Let $N \ge 2$, $n/2 < s_k \le n$, $n/s_k < p_k < \infty$, $k = 1, \dots, N$, $1/p_1 + \cdots + 1/p_N = 1/p$, and set $q_k = n/s_k$. Jiao [9] proved that if $\vec{w} \in A_{\vec{P}/\vec{Q}}$, then

$$\|T_m\|_{L^{p_1}(w_1)\times\cdots\times L^{p_N}(w_N)\to L^{p}(\nu_{\vec{w}})} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{W^{s_1+\cdots+s_N}(\mathbb{R}^{Nn})}.$$

In the case $q_1 = \cdots = q_N = Nn/s$, this coincides with the results of [1, 13] (see (1.5)). However, in the same way as in the proof of Theorem 1.1, we can prove that the estimate

$$\|T_m\|_{L^{p_1}(w_1)\times\cdots\times L^{p_N}(w_N)\to L^{p}(\nu_{\vec{w}})} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(s_1,\dots,s_N)}(\mathbb{R}^{Nn})}$$

does not in general hold for $\vec{w} \in A_{\vec{P}/\vec{Q}}$ (but this estimate holds for $\vec{w} \in \prod_{k=1}^N A_{p_k/q_k}$, see (1.4) and also [3, Therem 6.2] for general case). Indeed, as for (3.1) and (3.2), we can choose α_1 and α_2 so that $\alpha_1/p_1 + \alpha_2/p_2 > -n/p$, $\alpha_k/p_k < s_k - n/p_k$, $k = 1, 2$, and $\alpha_1/p_1 < -n/p_1 - s_1 + n/2$. Then $(|x|^{\alpha_1}, |x|^{\alpha_2}, 1, \dots, 1) \in A_{\vec{P}/\vec{Q}}$, and the rest of the proof is similar.

References

[1] T.A. Bui and X.T. Duong, Weighted norm inequalities for multilinear operators and applications to multilinear Fourier multipliers, Bull. Sci. Math. 137 (2013), 63–75.

[2] R. Coifman and Y. Meyer, Au delà des opérateurs pseudo-différentiels, Astérisque 57 (1978), 1–185.

[3] M. Fujita and N. Tomita, Weighted norm inequalities for multilinear Fourier multipliers, Trans. Amer. Math. Soc. 364 (2012), 6335–6353.
[4] L. Grafakos, Modern Fourier Analysis, Second edition, Graduate Texts in Mathematics 250, Springer, New York, 2009.
[5] L. Grafakos, A. Miyachi and N. Tomita, On multilinear Fourier multipliers of limited smoothness, Canad. J. Math. 65 (2013), 299–330.
[6] L. Grafakos and Z. Si, The Hörmander multiplier theorem for multilinear operators, J. Reine Angew. Math. 668 (2012), 133–147.
[7] L. Grafakos and R. Torres, Multilinear Calderón-Zygmund theory, Adv. Math. 165 (2002), 124–164.
[8] G. Hu and C.-C. Lin, Weighted norm inequalities for multilinear singular integral operators and applications, arXiv:1208.6346 (2012).
[9] Y. Jiao, A weighted norm inequality for multilinear Fourier multiplier operator, Math. Inequal. Appl., to appear.
[10] C. Kenig and E.M. Stein, Multilinear estimates and fractional integration, Math. Res. Lett. 6 (1999), 1–15.
[11] D. Kurtz and R. Wheeden, Results on weighted norm inequalities for multipliers, Trans. Amer. Math. Soc. 255 (1979), 343–362.
[12] A. Lerner, S. Ombrosi, C. Pérez, R. Torres and R. Trujillo-González, New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory, Adv. Math. 220 (2009), 1222–1264.
[13] K. Li and W. Sun, Weighted estimates for multilinear Fourier multipliers, arXiv:1207.5111 (2012).
[14] W. Li, Q. Xue and K. Yabuta, Weighted version of Carleson measure and multilinear Fourier multiplier, Forum Math., to appear.
[15] A. Miyachi and N. Tomita, Minimal smoothness conditions for bilinear Fourier multipliers, Rev. Mat. Iberoam. 29 (2013), 495–530.
[16] N. Tomita, A Hörmander type multiplier theorem for multilinear operators, J. Funct. Anal. 259 (2010), 2028–2044.

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